been evaluated. The maximum slope thickness L is obtained from the relation

$$L = (\rho_2 - \rho_1)/(d\rho/dx)_{max} \tag{2}$$

where ρ_1 is the density ahead of the shock, ρ_2 the density behind the shock and $(d\rho/dx)_{max}$ the maximum slope of the density profile. The density ρ_2 is given by the Rankine-Hugoniot conditions. In Fig. 5 the experimental results for the reciprocal shock wave thickness are shown as a function of the shock Mach number M together with theoretical results based on the Navier-Stokes equations (curve 1) and on Mott-Smith's bimodal theory (curve 2). Both curves have been taken from Linzer and Hornig's paper (curves 1 and 2a in Fig. 4 of Ref. 4). The shaded region represents the location of experimental results obtained by Talbot[5,6] and Sherman with a free-molecule probe and by Linzer and Hornig[4] with the optical reflectivity method. It can be seen from Fig. 5 that the present experiments are in agreement with these experimental results.

During the test runs with the new device a few experiments have been made with a driver pressure of 1 atm and a test gas pressure of 0.1 torr. In this case the pressure difference between high-pressure section and low-pressure section was high enough to break a diaphragm of hostaphan sheet. It was therefore for these conditions possible to start the shock tube flow once with the conventional diaphragm technique and the next time with the pneumatic valve. The density histories obtained with both methods were practically identical.

The main advantages of the instrument which has been described may be summarized as follows:

1) The new device opens practically independently of the pressure difference between driver section and driven section. It becomes therefore possible to use very low driver pressures in order to obtain low shock Mach numbers.

2) The device can be opened always at exactly the same driver pressure. This means that a very good reproduction of the state of the shocked gas is possible.

3) One can always use the same gas as test gas and as driver gas because the desired shock Mach number can be adjusted with the driver pressure. This means that the driver section delivers no foreign gas into the test section during a run. The pressure increase by a shot was only a few torr in our shock tube because of the low driver pressures and the dumping tank with a volume of 1200 liter so that preparing the shock tube for the next run consisted in lowering the pressure in the driven section by a few torr and admitting the driver gas. The auxiliary diaphragm D can be exchanged very fast, because there are no particular vacuum requirements for chambers A and B. With the shock tube described here it was possible to make a run every 2 min. An additional advantage of the low driver pressures which could be used here was the reduction of the gas flow from the shock tube into the electron gun. As a consequence the lifetime of the cathode was appreciably increased.

4) The instrument may be very useful for the investigation of relaxation effects in weak shock waves which occur in the atmosphere as sonic bangs.[7]

References

[1] Bradley, J. N., *Shock Waves in Chemistry and Physics*, Wiley, New York, 1962, p. 161.

[2] Schultz-Grunow, F. and Frohn, A., "Density Distribution in Shock Waves Traveling in Rarefied Monatomic Gases," *Rarefied Gas Dynamics*, Vol. I, edited by J. H. de Leeuw, Academic Press, New York, 1965, pp. 250–264.

[3] Mirels, H., "Test Time in Low Pressure Shock Tubes," *Physics of Fluids*, Vol. 6, 1963, pp. 1201–1214.

[4] Linzer, M. and Hornig, D. F., "Structure of Shock Fronts in Argon and Nitrogen," *Physics of Fluids*, Vol. 6, 1963, pp. 1661–1668.

[5] Sherman, F. S., "Study of Shock Wave Structure and Relaxation Phenomena in Gases," TN 3298, 1955, NACA.

[6] Talbot, L. and Sherman, F. S., "Structure of Weak Shock Waves in a Monatomic Gas," Memo 12-14-58W, 1959, NASA.

[7] Hodgson, J. P., "Vibrational Relaxation Effects in Weak Shock Waves in Air and the Structure of Sonic Bangs," *Journal of Fluid Mechanics*, Vol. 58, Pt. 1, 1973, pp. 187–196.

A Method of Static Analysis of Shallow Shells

R. JONES*
Swinburne College of Technology, Victoria, Australia
AND
J. MAZUMDAR†
The University of Adelaide, South Australia

Introduction

THE present work may be considered as a sequel to earlier work[1–5] where a method for a large class of boundary value problems associated with the bending, buckling and vibration analysis of elastic plates and membranes of arbitrary shape was developed. These papers introduced the concept of "Lines of Equal Deflection," i.e., contour lines which are obtained by intersecting the bent plate by planes parallel to the original plane of the plate. The present Note deals with the same method as applied to shallow shells with small deflections. Many of the considerations reported in this Note are treated in greater detail in the thesis of Jones.[6] As an illustration of the method, a technically interesting example of a shallow elliptical dome is examined.

An Account of the Method

Consider an elastic isotropic shallow shell of thickness h subject to a continuously distributed normal load. Let the equation of the middle surface of the shell referred to a system of orthogonal coordinates xyz be given by

$$z = x^2/2R_x + xy/R_{xy} + y^2/2R_y \tag{1}$$

When the shell is acted upon by a transverse load $q(x,y)$ then the intersections between the deflected surface and the parallels $z = \text{const}$ yield contours which after projection onto the $z = 0$ surface are the level curves called the Lines of Equal Deflection. Denote the family of such curves by $u(x,y) = \text{const}$. If the boundary of the shell is subjected to any combination of clamping and simple support, then clearly the boundary will belong to the family of lines of equal deflection and without loss in generality we may consider that $u = 0$ on the boundary. It is clear that the lines of equal deflection form a system of nonintersecting closed curves starting from the closed boundary as one of the lines.

Consider the equilibrium of an element of the shell bounded by any contour line of constant deflection. The conditions for the equilibrium of an element of the shell require that the sum of the moments about the tangent line at any point to the curve $u(x,y) = \text{const}$ of all the forces acting on the element, and the sum of all the forces normal to the plane $z = 0$ to vanish. Therefore proceeding exactly the same way as in Ref. 1, we obtain

$$\sum M = \mathbf{n}_o \cdot \oint M_n \mathbf{n} ds + \mathbf{n}_o \cdot \oint V_n \mathbf{r}_o ds - \mathbf{n}_o \cdot \iint \left[q - \frac{N_x}{R_x} - \frac{N_y}{R_y} - \frac{2N_{xy}}{R_{xy}} \right] \mathbf{r} d\Omega = 0 \tag{2}$$

and

$$\sum Z = \oint V_n ds - \iint \left[q - \frac{N_x}{R_x} - \frac{N_y}{R_y} - \frac{2N_{xy}}{R_{xy}} \right] d\Omega = 0 \tag{3}$$

where (x_o, y_o) is a fixed point on the line $u = \text{const}$, $\mathbf{n}$ and $\mathbf{n}_o$ denote the unit vectors normal to this line at any arbitrary point (x,y) and at the fixed point (x_o, y_o), respectively, $\mathbf{r}$ and $\mathbf{r}_o$ denote the position vectors from the fixed point (x_o, y_o) to any arbitrary point inside the contour and on the contour, respectively.

Received June 8, 1973; revision received February 25, 1974.
Index category: Structural Static Analysis.

* Lecturer, Department of Civil Engineering.
† Senior Lecturer, Department of Applied Mathematics.

Using the well-known expressions for the moments and shearing forces[7] and assuming the membrane forces N_x, N_y, and N_{xy} are determined by

$$N_x = \partial^2\phi/\partial y^2, \quad N_y = \partial^2\phi/\partial x^2, \quad N_{xy} = -\partial^2\phi/\partial x\,\partial y \tag{4}$$

where $\phi(x, y)$ is the stress function, Eqs. (2) and (3) reduce to

$$\mathbf{n}_o \frac{d^2w}{du^2}\oint P\,\mathbf{n}ds + \mathbf{n}_o \frac{dw}{du}\oint Q\,\mathbf{n}ds + \mathbf{n}_o \frac{d^3w}{du^3}\oint R\,\mathbf{r}_o\,ds + \mathbf{n}_o \frac{d^2w}{du^2}\oint F\,\mathbf{r}_o\,ds + \mathbf{n}_o \frac{dw}{du}\oint G\,\mathbf{r}_o\,ds - \mathbf{n}_o \iint\left[q - K_x\frac{\partial^2\phi}{\partial y^2} - K_y\frac{\partial^2\phi}{\partial x^2} + 2K_{xy}\frac{\partial^2\phi}{\partial x\,\partial y}\right]\mathbf{r}d\Omega = 0 \tag{5}$$

and

$$\frac{d^3w}{du^3}\oint R ds + \frac{d^2w}{du^2}\oint F ds + \frac{dw}{du}\oint G ds - \iint\left[q - K_x\frac{\partial^2\phi}{\partial y^2} - K_y\frac{\partial^2\phi}{\partial x^2} + 2K_{xy}\frac{\partial^2\phi}{\partial x\,\partial y}\right]d\Omega = 0 \tag{6}$$

where the expressions for P, Q, R, F, and G are given in Ref. 1. The condition for the continuity of deformation in this case reduces to

$$\nabla^4\phi = Eh\left[K_x\frac{\partial^2 w}{\partial y^2} + K_y\frac{\partial^2 w}{\partial x^2} - 2K_{xy}\frac{\partial^2 w}{\partial x\,\partial y}\right] \tag{7}$$

where K_x, K_y, and K_{xy} denote the curvatures at a point.

It is to be noted that the moment Eq. (5) and the force equation (6) are identical when $u(x, y) = \text{const}$ is the correct form of the lines of constant deflection. Consequently our problem reduces to solving Eqs. (6) and (7) for ϕ and w, using the exact expression for the lines of constant deflection.

Since the geometry of the shell is considered to be Euclidean and since we are using Reissner's assumptions regarding shallow shells,[8] our first approach is to assume the deflection contours at least in the case of uniform loading to be as for the corresponding flat plate problem ($K_x = K_y = K_{xy} = 0$). The question of determination of contour curves in the most general case however has been discussed in detail in Ref. 6.

Illustration

As an illustration, consider the bending of a shallow dome of nonzero gaussian curvature upon an elliptic base. The geometry of the shell being as described in Fig. 1, where the origin of coordinates is taken at the center, and where the edges of the shell are rigidly clamped. Supposing that the shell is under uniformly distributed loading then the lines of constant deflection as for the corresponding flat plate problem will be a family of similar and similarly situated ellipses. Consequently we consider

$$u(x, y) = 1 - x^2/a^2 - y^2/b^2 \tag{8}$$

Calculating the values of R, F, and G as carried out in Ref. 1 and performing the necessary integrations, the differential equation (6) finally reduces to

$$(1-u)^2\frac{d^3w}{du^3} - 2(1-u)\frac{d^2w}{du^2} - \delta\iint\left(K_x\frac{\partial^2\phi}{\partial y^2} + K_y\frac{\partial^2\phi}{\partial x^2}\right)d\Omega + q_1(1-u) = 0 \tag{9}$$

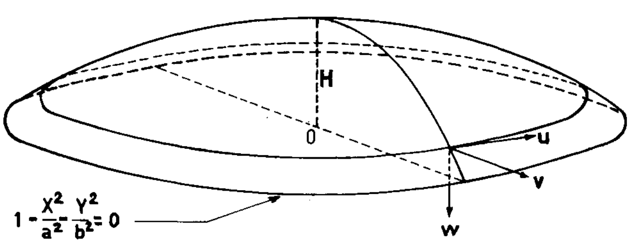

Fig. 1 Shallow elliptical dome.

where

$$q_1 = qa^4b^4/2D(3a^4 + 2a^2b^2 + 3b^4) \tag{10}$$

and

$$\delta = a^3b^3/2D(3a^4 + 3b^4 + 2a^2b^2)\pi \tag{11}$$

Furthermore, integrating Eq. (7) over the region under consideration, we obtain

$$\iint\left[\nabla^4\phi - Eh\left(K_x\frac{\partial^2\phi}{\partial y^2} + K_y\frac{\partial^2\phi}{\partial x^2}\right)\right]d\Omega = 0 \tag{12}$$

which after applying Green's theorem reduces to

$$\oint\nabla(\nabla^2\phi)\cdot\mathbf{n}ds - Eh\oint\left(K_x\frac{\partial w}{\partial y}\frac{u_y}{(t)^{1/2}} + K_y\frac{\partial w}{\partial x}\frac{u_x}{(t)^{1/2}}\right)ds = 0 \tag{13}$$

Assuming that ϕ is a function of u, Eq. (13) finally reduces to

$$\frac{d^3\phi}{du^3}\oint R\,ds + \frac{d^2\phi}{du^2}\oint F\,ds + \frac{d\phi}{du}\oint G\,ds - EhD\frac{dw}{du}\oint\left(\frac{K_x u_y^2}{(t)^{1/2}} + \frac{K_y u_x^2}{(t)^{1/2}}\right)ds = 0 \tag{14}$$

which after evaluating the contour integrals reduces to

$$(1-u)^2\frac{d^3\phi}{du^3} - 2(1-u)\frac{d^2\phi}{du^2} + Eh\gamma(1-u)\frac{dw}{du} = 0 \tag{15}$$

where

$$\gamma = (K_x/b^2 + K_y/a^2)a^4b^4/(3a^4 + 2a^2b^2 + 3b^4) \tag{16}$$

The first integral of (15) leads to

$$(d/du)[(1-u)(d\phi/du)] = Eh\gamma w + C \tag{17}$$

where C is an arbitrary constant. Using the abovementioned relation to Eq. (9) an ordinary differential equation in w results

$$(1-u)^2\frac{d^4w}{du^4} - 4(1-u)\frac{d^3w}{du^3} + 2\frac{d^2w}{du^2} + \frac{Eh\gamma^2 w}{D} + \frac{C\gamma}{D} - q_1 = 0 \tag{18}$$

a particular solution of which is

$$w = (q_1 D/Eh\gamma^2) - (C/Eh\gamma) \tag{19}$$

The general solution to the homogeneous equation corresponding to Eq. (18), is given by

$$w = w_1 + w_2, \tag{20}$$

where

$$w_1 = A_1J_0(kf) + A_2Y_0(kf) \tag{21}$$

and

$$w_2 = A_3I_0(kf) + A_4K_0(kf) \tag{22}$$

Here

$$f^2 = 1 - u, \quad k^2 = i4\lambda, \quad \text{and} \quad \lambda^2 = Eh\gamma^2/D \tag{23}$$

Consequently if we write

$$I_0[(i)^{1/2}x] = \mathrm{Ber}(x) + i\,\mathrm{Bei}(x) \tag{24}$$

and

$$K_0[(i)^{1/2}x] = \mathrm{Ker}(x) + i\,\mathrm{Kei}(x) \tag{25}$$

and use the requirement of finite deflection at the center $f = 0$, then

$$w = B_1\,\mathrm{Ber}(\omega f) + B_2\,\mathrm{Bei}(\omega f) + \frac{q_1 D}{E\gamma^2 h} - \frac{C}{EH\gamma} \tag{26}$$

where

$$\omega^2 = 4\lambda \tag{27}$$

and B_1 and B_2 are arbitrary constants.

The stress function ϕ may now be determined from Eq. (17) which in terms of the new variable f yields

$$\phi = \frac{4Eh\gamma}{\omega^2}[B_1\,\mathrm{Bei}(\omega f) - B_2\,\mathrm{Ber}(\omega f)] + \frac{q_1 Df^2}{\gamma} \tag{28}$$

The three unknown constants B_1, B_2, and C can now be determined by applying the clamping conditions

$$w\Big|_{f=1} = \frac{dw}{df}\Big|_{f=1} = \left[\frac{d^2\phi}{df^2} - \frac{\mu}{f}\frac{d\phi}{df}\right]_{f=1} = 0 \tag{29}$$

The last boundary condition representing the vanishing of the circumferential strain was proposed by Reissner[8] and Gradowczyk.[9]

It is interesting to note that if we put $a = b$, and $K_x = K_y = K$ the solution coincides exactly with that obtained by Gradowczyk[9] for a clamped spherical dome with curvature K. Further in the limiting case as K_x and K_y tend to zero we obtain

$$\lim_{K_x, K_y \to 0} W = q_1(1-f^2)^2/4 = q_1(1-x^2/a^2 - y^2/b^2)^2/4 \tag{30}$$

which is the exact expression for the bending of a clamped, uniformly loaded elliptic plate.

References

[1] Mazumdar, J., "A Method for Solving Problems of Elastic Plates of Arbitrary Shape," *Journal of Australian Mathematical Society*, Vol. 11, 1970, pp. 95–112.

[2] Mazumdar, J., "Transverse Vibration of Elastic Plates by the Method of Constant Deflection Lines," *Journal of Sound and Vibration*, Vol. 18, 1971, pp. 147–155.

[3] Mazumdar, J., "Buckling of Elastic Plates by the Method of Constant Deflection Lines," *Journal of Australian Mathematical Society*, Vol. 13, 1972, pp. 91–103.

[4] Mazumdar, J., "Transverse Vibration of Membranes of Arbitrary Shape by the Method of Constant Deflection Contours," *Journal of Sound and Vibration*, Vol. 27, 1973, pp. 47–57.

[5] Mazumdar, J. and Jones, R., "A Simplified Approach to the Analysis of Large Deflections of Plates," *Journal of Applied Mechanics* (in press).

[6] Jones, R., "Application of the Method of Constant Deflection Contour Lines to Elastic Plates and Shell Problems," Ph.D. thesis, 1973, The University of Adelaide, Adelaide, South Australia.

[7] Timoshenko, S. P. and Woinowsky-Krieger, S., *Theory of Plates and Shells*, McGraw-Hill, New York, 1959.

[8] Reissner, E., "Stresses and Small Displacements of Shallow Spherical Shells," *Journal of Mathematics and Physics*, Vol. 25, 1946, pp. 80–85, 279–300.

[9] Gradowczyk, M. H., "Some Remarks on the Theory of Shallow Spherical Shells," *Ingenieur Archiv.*, Vol. 32, 1963, pp. 297–303.

Note on Boundary Layer in a Dusty Gas

K. M. Chakrabarti*

Regional Engineering College, Durgapur, West Bengal, India

Introduction

SPROULL[1] has observed that adding dust to air flowing in turbulent motion through a pipe appreciably reduces the resistance coefficient. Saffman[2] discussed in detail the stability of laminar flow of a dusty gas. Michael[3,4] has investigated the plane parallel flow of a dusty gas. It is supposed that the dust particles are uniform in size and shape, so that the stability of the flow is conserved. This Note initiates boundary-layer theory for a dusty gas past an infinite plate in the primary stages of the motion (Rayleigh's Case). We have discussed the effect on the boundary layer of the potential flow of a gas in presence of dust particles. The density of the dust material is allowed to be large compared with the gas density, so that the mass concentration of the dust f is small. The expression for local skin friction due to the dusty gas has been deduced and has been compared with that for the case of a clean gas. It is seen that the effect of dust reduces the local skin friction. This reduction tends to diminish as t increases. However, this analysis is valid only at the initial stage of a boundary-layer development, as in the case of Lord Rayleigh.

Received August 9, 1973; revision received December 12, 1973. The author is very grateful to S. K. Ghoshal of Jadavpur University for his guidance and helpful suggestions in preparing this Note.

Index categories: Boundary Layers and Convective Heat Transfer—Laminar; Nonsteady Aerodynamics.

* Lecturer in Mathematics.

Formulation of Equations and Solutions

Let the velocity and number density of the dust particles be described by the fields $\mathbf{v}(\mathbf{x}, t)$ and $N(\mathbf{x}, t)$. Assuming that the particles of the dust are small enough to make the Stokes law of resistance between the dust particles and gas appropriate, the equations of motion become

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\operatorname{grad} p + \nu\nabla^2\mathbf{u} + \frac{kN}{\rho}(\mathbf{v}-\mathbf{u}) \tag{1}$$

$$m\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = k(\mathbf{u}-\mathbf{v}) \tag{2}$$

$$\operatorname{div}\mathbf{u} = 0 \tag{3}$$

$$\frac{\partial N}{\partial t} + \operatorname{div} N\mathbf{v} = 0 \tag{4}$$

In these equations, $\mathbf{u}$ and $\mathbf{v}$ are the velocities of the gas and dust particles, respectively, k being the Stokes resistance coefficient which, for spherical particles of radius a, is $6\pi\mu a$, μ the viscosity of the gas, ρ the density of the gas, p the gas pressure, and $\tau = m/k$ the relaxation time of the dust particles where m is the mass of the dust particles.

In the present case, we have assumed that the dust is uniformly distributed in the gas and the motion is induced by the potential flow $u_\infty(x,t) = U(x)$. Let $\mathbf{u} = u(y,t)\mathbf{x}$, $\mathbf{v} = v(y,t)\mathbf{x}$, and Eq. (4) is satisfied with $N = N_o$ (a constant) throughout the motion. Equation (3) is identically satisfied. And it remains to solve Eqs. (1) and (2), which can be written as

$$\partial u/\partial t = \nu(\partial^2 u/\partial y^2) + (kN_o/\rho)(v-u) \tag{5}$$

$$\tau(\partial v/\partial t) = u - v \tag{6}$$

We shall work in terms of the dimensionless time variable $\bar{t} = t/\tau$ and the dimensionless length $\bar{y} = y/(\nu\tau)^{1/2}$, in which case the equations become (dropping bars)

$$\partial u/\partial t = (\partial^2 u/\partial y^2) + f(v-u) \tag{7}$$

$$\partial v/\partial t = u - v \tag{8}$$

where $f = N_o m/\rho$ is the mass concentration of dust.

Let us express u and v as follows:

$$u = u_o + f u_1 + f^2 u_2 + \cdots \tag{9}$$

$$v = v_o + f v_1 + f^2 v_2 + \cdots \tag{10}$$

The first and second approximations of relations (9) and (10) are taken, respectively, as $u = u_o$, $v = v_o$ (first approximation); $u = u_o + f u_1$, $v = v_o + f v_1$ (second approximation). Taking the first approximation, Eqs. (7) and (8) become

$$\partial u_o/\partial t = \partial^2 u_o/\partial y^2 \tag{11}$$

$$\partial v_o/\partial t = u_o - v_o \tag{12}$$

with the boundary conditions

$$y = 0,\ u_o = 0; \quad y = \infty,\ u_o = U(x)$$

and initial conditions $t = 0$, $u_o = 0$, $v_o = 0$.

We shall now try to solve Eqs. (11) and (12) with the help of the abovementioned boundary and initial conditions. The solution of Eq. (11) is

$$u_o(x,y,t) = U(x)\cdot\frac{2}{(\pi)^{1/2}}\int_o^\eta e^{-\eta^2} d\eta \tag{13}$$

Where

$$\eta = y/2(t)^{1/2}$$

with the help of Eq. (13), Eq. (12) becomes

$$\frac{\partial v_o}{\partial t} + v_o = \frac{2}{(\pi)^{1/2}} U(x)\int_o^\eta e^{-\eta^2} d\eta \tag{14}$$